JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 17, No. 1, January–February 1994

Disturbance Attenuation and H^∞ Optimization with Linear Output Feedback Control

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Recently, many important results have been established for solving the following problem in linear system theory: Design a dynamic output feedback compensator such that the gain of the closed-loop transfer function is minimized over all frequencies. This paper considers disturbance attenuation and H^∞ optimization with linear static output feedback control. Under certain minimum phase assumption on the open-loop plant and the condition that the open-loop system is stabilizable using high gain feedback, the achievable disturbance attenuation level with constraint linear static output feedback is obtained. A synthesis procedure to achieve prespecified disturbance attenuation is developed and illustrated by means of an example.

I. Introduction

THE disturbance attenuation problem is concerned with designing a feedback control law that ensures that the effect of the disturbance acting on a linear system is reduced to an acceptable level. H^∞ optimal control is one specific version of this problem in which the disturbance consists of energy bounded signals and the design objective is to minimize the gain from disturbance input to regulated output over all frequencies. Such a problem also arises in stability robustness of feedback systems (see Refs. 6 and 11). Complete solutions are available when linear dynamic output feedback controls are used (see Refs. 4 and 17 for details).

This paper studies disturbance attenuation and H^∞ optimization using linear static output feedback control. It is assumed that the open-loop system can be stabilized with high gain feedback, i.e., a static output feedback gain exists such that the closed-loop system is asymptotically stable as long as the feedback gain is sufficiently high. We will make this assumption more specific in the next section. Furthermore, we will restrict our feedback compensators without memory to those solvable from state feedback laws, which are termed as constraint linear static output feedback controls.

In a previous paper,[5] we established a necessary and sufficient condition for the existence of linear quadratic optimal control when linear static output feedback is used. Disturbance attenuation and H^∞ optimization using linear static output feedback is also studied for a special class of systems. In this paper, we extend the disturbance attenuation problem as formulated in Ref. 5 to a more general class of linear systems. It will be shown that under certain minimum phase condition and the condition that the open-loop system can be stabilized using high gain output feedback, optimal control with constraint linear static output feedback can be achieved asymptotically. Moreover, the achievable disturbance attenuation level with constraint linear static output feedback is obtained, and a synthesis procedure to achieve prespecified disturbance attenuation is developed.

The results reported in this paper have applications to the problem of stabilization of uncertain systems.[6,11] It is also hoped that the results in this paper complement the existing research reported in Refs. 3, 7, and 9, in which linear static output feedback control is studied extensively.

II. Preliminary

The system under consideration is described by the following state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t) + Dd(t), \qquad y(t) = Cx(t)$$
$$w(t) = Ex(t) \tag{1}$$

where $x(t) \in \boldsymbol{R}^n$ is the state, $d(t) \in \boldsymbol{R}^p$ is the disturbance input, $u(t) \in \boldsymbol{R}^m$ is the control input, $y(t) \in \boldsymbol{R}^l$ is the measured output, and $w(t) \in \boldsymbol{R}^q$ is the controlled output.

Assumptions: In the sequel, it will be assumed that system (1) satisfies the following assumptions:

1) The number of control inputs is equal to the number of measured outputs: $l = m$.

2) The high frequency gain from the control input to the measured output is nonsingular; i.e., $\det(CB) \neq 0$.

3) The transfer function $C(sI - A)^{-1}B$ is strict minimum phase; i.e., for all $\mathrm{Re}(s) \geq 0$,

$$\mathrm{rank}\begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix} = n + m$$

Although the assumptions on the plant model are restrictive, the results reported in the next section are often applicable to a more general class of systems. For instance, if the plant model is minimum phase but does not satisfy assumptions 1 and 2, we can first apply the pre- or postcompensator as discussed in Ref. 13 to square down the plant and keep the minimum phase property and next approximate it with a plant model that meets the preceding assumptions.

Remark: Without loss of generality, it is assumed that the input matrix is in the special form

$$B^T = [I \quad 0] \tag{2}$$

Otherwise, it is always possible to find a similarity transform such that Eq. (2) is true. Also note that assumptions 1–3 are quite similar to those used in adaptive control.

The preceding assumptions imply the existence of a feedback gain $K \in \boldsymbol{R}^{m \times m}$ and a constant $\rho^* > 0$ such that the

Received July 27, 1992; revision received Feb. 10, 1993; accepted for publication Feb. 19, 1993.

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matrix $A - \rho BKC$ has all its eigenvalues on the open left half-plane for all $\rho \geq \rho^*$ (see Ref. 8 for more details). Hence, the open-loop system defined in Eq. (1) can be stabilized with high gain output feedback control.

As indicated in Ref. 5, if we partition the matrices A and C as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad C = [C_1 \quad C_1] \tag{3}$$

where A_{11} and C_1 in $\boldsymbol{R}^{m \times m}$. Then, by assumptions 1-3, C_1 is nonsingular, and the matrix

$$F = A_{22} - A_{21} C_1^{-1} C_2$$

is stable; i.e., all of the eigenvalues of F are on the open left half-plane.

Definition: Let the constant $\sigma > 0$ be given. Then the system (1) can be stabilized via a linear static compensator with (strict) disturbance attenuation σ if there exists a constant matrix K of size $m \times m$ such that the following conditions are satisfied: 1) the closed-loop system is internally stable; and 2) the transfer function matrix for the closed-loop system

$$T_{wd}(s) = E(sI - A + BKC)^{-1} D \tag{4}$$

satisfies the bound $\|T_{wd}\|_\infty \leq \sigma$ (or $\|T_{wd}\|_\infty < \sigma$ for strict disturbance attenuation) where $\|T_{wd}\|_\infty$ is defined by

$$\|T_{wd}\|_\infty = \sup\{\sqrt{\lambda_{\max}[T_{wd}^T(-j\omega) T_{wd}(j\omega)]} : \omega \in \boldsymbol{R}\} \tag{5}$$

Note that if the system (1) can be stabilized with disturbance attenuation σ, then the closed-loop system is internally stable. The following result was established in Ref 5.

Lemma 1[5]: Let the system be given as in system (1). Assume that the realization $\{A, B, C\}$ is both stabilized and detectable and satisfies assumptions 1-3. Assume further that $D = B\Omega$ for some Ω. Then, for any $\delta > 0$, there exists a matrix $K \in \boldsymbol{R}^{m \times m}$ such that the closed-loop system as in Eq. (4) is stabilized with attenuation δ, i.e., $\|T_{wd}\|_\infty \leq \delta$.

Clearly, the preceding result implies that the infimum of disturbance attenuation level using linear static output feedback is zero when the column space of the D matrix is contained in the column space of the B matrix. However, no result is available if this condition is violated. In this paper we will explore this class of disturbance attenuation problem in more detail. The next result can be found in Ref. 6.

Lemma 2[6]: Suppose that the state vector of the system (1) is available for measurement so that the state feedback $u(t) = Fx(t)$, $F \in \boldsymbol{R}^{m \times n}$, can be used and the realization pair $\{A, B\}$ can be stabilized. Then, the system (1) is internally stabilized with strict attenuation σ for some $F \in \boldsymbol{R}^{m \times n}$ if and only if the algebraic Riccati equation

$$A^T P + PA - (1/\epsilon) PBB^T P + (1/\sigma^2) PDD^T P + E^T E + \epsilon Q = 0 \tag{6}$$

has a positive definite solution P for some scalar $\epsilon > 0$ and some positive definite matrix Q of size $n \times n$.

The choice of the matrix Q does not affect the solvability of the algebraic Riccati equation.[6] Furthermore, with $u(t) = -(1/2\epsilon) B^T P x(t)$, the closed-loop system is internally stabilized with strict disturbance attenuation σ. If the state $x(t)$ is not available for measurement, then the linear static output feedback control $u(t) = -Ky(t)$ can be solved from

$$KC = (1/2\epsilon)\hat{K}C = (1/2\epsilon) B^T P \tag{7}$$

with $K \in \boldsymbol{R}^{m \times m}$ and P obtained from Eq. (6). Clearly, if Eqs. (6) and (7) admit solutions P and K for some $\epsilon > 0$ and some positive definite matrix Q, then the output feedback control $u(t) = -Ky(t)$ stabilizes the system (1) internally with strict disturbance attenuation σ. Since the algebraic Riccati equation is a powerful synthesis tool for multivariable dynamic feedback system design, it will be interesting to obtain an optimal solution among all linear static output feedback controls $u(t) = -Ky(t)$ solved from Eqs. (6) and (7) in terms of disturbance attenuation. That is, if we define the solution set

$$\boldsymbol{K} = \{K : K \text{ is solved from Eqs. (6) and (7), which internally stabilizes the system (1)}\} \tag{8}$$

then the intent is to obtain an optimal solution

$$\sigma^* = \inf\{\|T_{wd}\|_\infty : K \in \boldsymbol{K}\} \tag{9}$$

where $T_{wd}(s)$ is the transfer function for a closed-loop system as given in Eq. (4). Note that in general we have

$$\sigma^* = \inf\{\|T_{wd}\|_\infty : K \in \boldsymbol{K}\} \geq \inf\{\|T_{wd}\|_\infty : K \text{ stabilizing static output feedback}\} \tag{10}$$

At this point, it is unclear whether the preceding inequality can be replaced by an equality, but it will be shown in the next section that the optimal solution of Eq. (9) indeed can be obtained explicitly.

Remark: Very recently, a general result on static output feedback for H^∞ control has been presented in Ref. 14. However, only the existence part is resolved in Ref. 14. Although our result in the next section is quite restrictive, it provides a numerical algorithm to compute the necessary feedback gain, which is in contrast to that in Ref. 14. Future research should be directed to bridging the gap between the two results (see also Ref. 15).

III. Main Results

The following result will be used in the sequel:

Lemma 3[6,10,16]: Let $T(s) = H(sI - F)^{-1} G$ with F stable. Then, 1) given any positive definite matrix $Q \in \boldsymbol{R}^{n \times n}$ and scalar $\sigma > \|T\|_\infty$, there exist a positive definite matrix $P \in \boldsymbol{R}^{n \times n}$ and scalar $\epsilon > 0$ such that $F^T P + PF + (1/\sigma^2) PG^T GP + H^T H + \epsilon Q = 0$ and 2) if $\sigma < \|T\|_\infty$, then the preceding algebraic Riccati equation does not admit a solution P for any positive definite matrix Q and scalar $\epsilon > 0$.

Note that the preceding result does not require controllability and observability of the realization. The choice of Q does not affect the solvability of the preceding algebraic Riccati equation.

We have assumed, without loss of generality, that the control input matrix B is in special form (2). Therefore, the disturbance input matrix D can be decomposed as $D = D_1 + D_2$ where $B^T D_1 = 0$ and $D_2^T D_1 = 0$. It is not difficult to verify that such a decomposition has the following structure:

$$D_1^T = [0 \quad L^T] \qquad D_2^T = [\Omega^T \quad 0] \tag{11}$$

where $L \in \boldsymbol{R}^{(n-p) \times p}$ and $\Omega \in \boldsymbol{R}^{p \times p}$. The next two key results show how an optimal solution of σ^* in Eq. (10) can be found. Further, for any prespecified $\sigma > \sigma^*$, they show how we can obtain a stabilizing state feedback controller for the system (1) with disturbance attenuation σ.

Theorem 4: Suppose that the system (1) satisfies assumptions 1-3, and in addition $D^T B = 0$, i.e., $D_2 = 0$. Partition the matrices A, C, and E as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad C = [C_1 \quad C_2], \qquad E = [E_1 \quad E_2]$$

such that A_{11}, $C_1 \in \boldsymbol{R}^{m \times m}$ and $E_1 \in \boldsymbol{R}^{q \times m}$. Define the transfer function matrix

$$R(s) = (E_2 - E_1 C_1^{-1} C_2)(sI - F)^{-1} L, \quad \text{with } F = A_{22} - A_{21} C_1^{-1} C_2 \tag{12}$$

Then, 1) $R(s)$ is stable, and 2) for any $\sigma > \|R\|_\infty$, there exists a linear static output feedback control $u(t) = -Ky(t)$, $K \in \boldsymbol{K}$ [defined in Eq. (8)] such that the closed-loop system is internally stabilized with disturbance attenuation σ; i.e., $\|T_{wd}\|_\infty \le \sigma$.

Proof: Clearly, the matrix F as in Eq. (12) is stable by minimum phase assumption. To establish that for any $\sigma > \|R\|_\infty$ a linear static output feedback control $u(t) = -Ky(t)$ exists such that the $\|T_{wd}\|_\infty \le \sigma$, we need to show that there exists a positive definite matrix Q and a scalar $\epsilon > 0$ such that the matrix equations (6) and (7) have solutions $P = P^T > 0$ and $K \in \boldsymbol{R}^{m\times m}$. Note that since $\epsilon > 0$ and $Q > 0$, Eq. (6) is equivalent to the matrix inequality

$$P^{-1}A^T + AP^{-1} - (1/\epsilon)BB^T + (1/\sigma^2)DD^T + P^{-1}E^TEP^{-1} < 0$$

Denote $S = P^{-1}$ and define

$$\Phi = SA^T + AS - (1/\epsilon)BB^T + (1/\sigma^2)DD^T + SE^TES < 0 \quad (13)$$

We thus need to show that if $\sigma > \|R\|_\infty$, the preceding inequality admits a positive definite solution $S\ (= P^{-1})$ that satisfies Eq. (7) for some $K \in \boldsymbol{R}^{m\times m}$ with Φ some negative definite matrix. Now partition

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, \qquad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

conformably to matrix A as in Eq. (3). With these partitions, Eq. (7) can be written as

$$C_1S_{11} + C_2S_{21} = \hat{K}^{-1}, \qquad C_1S_{12} + C_2S_{22} = 0$$

and hence

$$S_{12} = -C_1^{-1}C_2S_{22} \quad (14)$$

Furthermore, with S_{12} so defined, Eq. (13) is equivalent to

$$\Phi_{11} = S_{11}A_{11}^T + A_{11}S_{11} + S_{12}A_{12}^T + A_{12}S_{21} - (I/\epsilon) + [S_{11} \quad S_{12}]E^TE[S_{11} \quad S_{12}]^T \quad (15)$$

$$\Phi_{22} = S_{22}(A_{22} - A_{21}C_1^{-1}C_2)^T + (A_{22} - A_{21}C_1^{-1}C_2)S_{22} + (1/\sigma^2)LL^T + S_{22}\hat{E}^T\hat{E}S_{22} \quad (16)$$

where $\hat{E} = E_2 - E_1C_1^{-1}C_2$, and

$$\Phi_{12} = S_{11}A_{21}^T + A_{11}S_{12} + S_{12}A_{22}^T + A_{12}S_{22} + [S_{11} \quad S_{12}]E^TE[S_{21} \quad S_{22}]^T \quad (17)$$

By Lemma 3, Eq. (16) has a positive definite solution S_{22} for some negative definite Φ_{22}, provided $\|\Phi_{22}\|$ is suitably small to satisfy $\|R\|_\infty < \sigma$. Hence, we can solve for S_{22} by selecting an appropriate Φ_{22}. Knowing S_{22}, we can solve for S_{12} from Eq. (14). Finally, we can obtain S_{11} by the inequality

$$S_{11} > S_{12}S_{22}^{-1}S_{21}$$

This choice of S_{11} guarantees that S is real, symmetric, and positive definite. Clearly, since S is positive definite, the solution $P = S^{-1}$ of Eq. (7) is also positive definite. Moreover, with P partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Eq. (7) can be rewritten as

$$\hat{K}C_1 = P_{11}, \qquad \hat{K}C_2 = P_{12}$$

from where it is clear that $P_{12} = P_{11}C_1^{-1}C_2$. The feedback gain $\hat{K}$ as in Eq. (7) can then be obtained as

$$\hat{K} = (C_1S_{11} + C_2S_{21})^{-1} = B^TC^T(CSC^T)^{-1} \quad (18)$$

Since S is positive definite, the existence of $\hat{K}$ is guaranteed. Next, we will show that with S and $\hat{K}$ as determined earlier, the matrix Φ is negative definite, provided $\epsilon > 0$ is sufficiently small. Indeed, Φ being negative definite is equivalent to

$$\Phi_{22} < 0, \qquad \Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{21} < 0$$

Note that when solving S_{22} in Eq. (16), Φ_{22} has been set as negative definite. Furthermore, from Eq. (17) it is clear that $\Phi_{12}\ (= \Phi_{21}^T)$ does not depend on ϵ. Therefore, from Eq. (15) we can see that the submatrix

$$\Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{21} + (I/\epsilon)$$

is independent of ϵ. Therefore, $\Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{21} < 0$ is ensured if in Eq. (15) $\epsilon > 0$ is sufficiently small. Hence, with output feedback control law

$$u(t) = -(1/2\epsilon)\hat{K}y(t) = -Ky(t)$$

the closed-loop system is internally stable with disturbance attenuation σ. The choice of Q satisfying Eq. (6) is given by $Q = -(1/\epsilon)S^{-1}\Phi S^{-1}$. This concludes the proof. □

The disturbance attenuation σ is characterized by the following bound on σ^*:

Corollary 5: Suppose all of the conditions in Theorem 4 hold, and the matrices B and D satisfy $B^TD = 0$ with B in the same special form (2). Then,

$$\sigma^* = \inf\{\|T_{wd}\|_\infty : K \in \boldsymbol{K}\} = \|R\|_\infty$$

where $T_{wd}(s)$ is the transfer function for a closed-loop system as given in Eq. (4) and $R(s)$ is the transfer function as given in Eq. (12).

Proof: In light of Theorem 4, for any $\sigma > \|R\|_\infty$, an output feedback control $u(t) = -Ky(t)$, $K \in \boldsymbol{K}$ exists that is defined as in Eq. (8), such that the closed-loop system is internally stabilized with disturbance attenuation σ. Hence, $\sigma^* \le \|R\|_\infty$. We claim that it is impossible for σ^* to be strictly smaller than $\|R\|_\infty$. Using contradiction, assume that $\sigma^* < \|R\|_\infty$. Then there exists $\sigma > 0$ such that $\sigma^* < \sigma < \|R\|_\infty$, and hence there exists a $K \in \boldsymbol{K}$ such that with $u(t) = -Ky(t)$ the closed-loop system is internally stable with strict disturbance attenuation σ. In light of the proof of Theorem 4, the matrix inequality (13) should have at least one positive definite solution S for some positive definite matrix Q and scalar $\epsilon > 0$. Clearly, the solvability of S amounts to the solvability of S_{22} as in Eq. (16). However, since $\sigma < \|R\|_\infty$, no positive definite solution S_{22} exists satisfying Eq. (16) in light of Lemma 3 and hence a contradiction exists. □

In Theorem 4 we considered the disturbance that was restricted to the column space of the input of the system. Next we generalize our results to arbitrary disturbances. The next result gives a complete solution for the disturbance attenuation problem using constraint linear static output feedback control. We again assume without loss of generality that the control input matrix B is in the special form of Eq. (2) and the disturbance input matrix is decomposed into $D = D_1 + D_2$, where D_1 and D_2 are in the form of Eq. (11). Hence $B^TD_1 = 0$ and $D_2^TD_1 = 0$. Note that for Theorem 4 it was assumed that $D_2 = 0$, whereas here that assumption has been removed.

Theorem 6: Assume that the system (1) satisfies assumptions 1-3. Define $R(s)$ as

$$R(s) = (E_2 - E_1C_1^{-1}C_2)(sI - F)^{-1}L,$$

$$\text{with } F = A_{22} - A_{21}C_1^{-1}C_2 \quad (19)$$

which is similar to Eq. (12), and matrices A, C, and E are partitioned in a manner similar to Theorem 4. Then, $R(s)$ is stable. Furthermore, for any $\sigma > \|R\|_\infty$, there exists an output feedback control $u(t) = -Ky(t)$, $K \in \boldsymbol{K}$ that is defined as in Eq. (8) such that the closed-loop system is internally stable with disturbance attenuation σ.

Proof: The stability of $R(s)$ is obvious. Consider the transfer function of the closed-loop system

$$T_{wd}(s) = E(sI - A + BKC)^{-1}D = T_1(s) + T_2(s)$$

with

$$T_1(s) = E(sI - A + BKC)^{-1}D_1$$

and

$$T_2(s) = E(sI - A + BKC)^{-1}D_2$$

We need to show that for any $\sigma > \|R\|_\infty$, there exists a positive definite matrix Q and a scalar $\epsilon > 0$ such that the matrix equations (6) and (7) have solutions $P = P^T > 0$ and $K \in \boldsymbol{R}^{m \times m}$. Indeed, since $\sigma > \|R\|_\infty$, there exists $\delta > 0$ such that $\sigma - \delta > \|R\|_\infty$. Define $\sigma_1 = \sigma - \delta$. By Theorem 4, there exists a matrix $K \in \boldsymbol{K}$ such that $T_2(s)$ is internally stable and $\|T_2\|_\infty \le \sigma_1$. Furthermore, $K = (1/2\epsilon)\hat{K}$, together with S, can be solved from Eqs. (15-18). With solutions S and $\hat{K}$, we can verify that

$$\hat{\Phi}_{22} = S_{22}(A_{22} - A_{21}C_1^{-1}C_2)^T + (A_{22} - A_{21}C_1^{-1}C_2)S_{22} + S_{22}\hat{E}^T\hat{E}S_{22}$$

is negative definite by Eq. (16). Furthermore,

$$\hat{\Phi}_{12} = S_{11}A_{21}^T + A_{11}S_{12} + S_{12}A_{22}^T + A_{12}S_{22} + [S_{11} \quad S_{12}]E^TE[S_{21} \quad S_{22}]^T$$

does not involve δ and ϵ. Hence,

$$\hat{\Phi}_{11} = S_{11}A_{11}^T + A_{11}S_{11} + S_{12}A_{12}^T + A_{12}S_{21} - \left(\frac{I}{\epsilon} - \frac{\Omega\Omega^T}{\delta^2}\right) + [S_{11} \quad S_{12}]E^TE[S_{11} \quad S_{12}]^T$$

and $\hat{\Phi}_{11} - \hat{\Phi}_{12}\hat{\Phi}_{22}^{-1}\hat{\Phi}_{21}$ can be made negative definite by taking sufficiently small ϵ. Therefore, there exists an $\epsilon^* > 0$ such that $\hat{\Phi}$ is negative definite. By the proof of Theorem 4, the preceding three equations imply that

$$\hat{\Phi} = SA^T + AS - (1/\epsilon^*)BB^T + (1/\delta^2)D_1D_1^T + SE^TES < 0$$

and hence $T_1(s)$ is internally stable, and in light of Lemma 2, $\|T_1\|_\infty \le \delta$. By setting $K = (1/2\epsilon^*)\hat{K}$, we thus have

$$\|T_{wd}\|_\infty \le \|T_1\|_\infty + \|T_2\|_\infty \le \delta + \sigma_1 = \sigma$$

which concludes the proof. □

Finally, we again assume without loss of generality that the control input matrix B is in the special form of Eq. (2) and the disturbance input matrix is decomposed into $D = D_1 + D_2$, where D_1 and D_2 are in the form of Eq. (11). Hence $B^TD_1 = 0$ and $D_2^TD_1 = 0$.

Corollary 7: Suppose that the system (1) satisfies assumptions 1-3. Define $R(s)$ as in Eq. (19). Then,

$$\sigma^* = \inf\{\|T_{wd}\|_\infty : K \in \boldsymbol{K}\} = \|R\|_\infty$$

where $\boldsymbol{K}$ is defined as in Eq. (8) and σ^* is defined in Eq. (9).

Proof: The results in Theorem 6 imply that $\sigma^* \le \|R\|_\infty$. We will show that $\sigma^* \ge \|R\|_\infty$ also. Using contradiction, suppose that $\sigma^* < \|R\|_\infty$. Then the algebraic Riccati equation (6) does not admit any solution for any given positive definite matrix Q and scalar $\epsilon > 0$. Indeed, the solvability of Eq. (6) is equivalent to the solvability of Eq. (13). Hence, if we take the (2,2) block of Eq. (13) with S_{12} as in Eq. (14) substituted in, we have

$$\Phi_{22} = S_{22}(A_{22} - A_{21}C_1^{-1}C_2)^T + (A_{22} - A_{21}C_1^{-1}C_2)S_{22} + (1/\sigma^2)LL^T + S_{22}\hat{E}^T\hat{E}S_{22}$$

By the hypothesis that $\sigma < \|R\|_\infty$, the preceding equation does not admit a solution S_{22} for any given $\Phi_{22} > 0$, in light of Lemma 3. Hence, a contradiction exists. □

It is worth mentioning that the results in this section not only establish the optimal solutions for Eq. (9) but also give a systematic procedure for synthesizing the feedback compensators. We summarize this section with the algorithm based on the results developed in the preceding sections.

Algorithm—Step 1: Input realization $\{A, B, C, D, E\}$ which satisfy assumptions 1-3.

Step 2: Find a similarity transformation matrix T such that TB is in the form of Eq. (2) and set

$$\{A, B, C, D, E\} = \{TAT^{-1}, TB, CT^{-1}, TD, ET^{-1}\}$$

Step 3: Partition A, C, and E as in Theorem 4 and decompose $D = D_1 + D_2$ with $D_1^T = [0 \quad L^T]$ and $D_2^T = [\Omega^T \quad 0]$ as in Theorem 6.

Step 4: Compute $\sigma^* = \|R\|_\infty$ where $R(s)$ is defined as in Eq. (19).

Step 5: For any $\sigma > \|R\|_\infty$, solve for $S = P^{-1}$ and $\hat{K}$ from Eqs. (15-18).

Step 6: Choose $\epsilon^* > 0$ such that the transfer function $T_{wd}(s)$ for a closed-loop system as in Eq. (4) is internally stable and attains disturbance attenuation $\|T_{wd}\|_\infty \le \sigma$ following the procedure next:

1) Set $\epsilon^* = 1$ (or other initial value for ϵ^*).

2) Set $K = \hat{K}/2\epsilon^*$ and test whether or not the stability of $A - BKC$ and $\|T_{wd}\|_\infty \le \sigma$ are both true.

3) If either one or both are not true, set $\epsilon^* := \epsilon^*/2$ and continue 2); otherwise, go to Step 7.

Step 7: Set the feedback gain as $K = \hat{K}/2\epsilon$ where $0 < \epsilon < \epsilon^*$. End.

We would like to point out that in designing feedback gain K, computing the matrix Φ as in Eq. (13) is not required. In fact, as shown in Theorems 4 and 6, the closed-loop system is internally stable and satisfies the disturbance attenuation $\sigma > \|R\|_\infty$ as long as we choose $\epsilon > 0$ sufficiently small after feedback gain K is solved from Eqs. (15-18). This result implies the existence of $\epsilon^* > 0$ such that for all strictly positive scalar $\epsilon \le \epsilon^*$, the closed-loop system is internally stable and $\|T_{wd}\|_\infty \le \sigma$. On the other hand, a small value of ϵ^* implies high gain in feedback, which may not be desirable because of the possible saturation. Hence, one would like to have ϵ^* as large as possible while achieving internal stability and disturbance attenuation. To find such ϵ^*, we note that the closed-loop transfer function $T_{wd}(s)$ is a function of ϵ since $K = \hat{K}/2\epsilon$. Therefore, the value of ϵ^* can be obtained by using a bisection search or other techniques[2] on parameter $\epsilon > 0$, which may refine step 6 of the preceding algorithm. That is, we use the bisection method to search the largest value ϵ^* such that the matrix $A - BKC$ has all of its eigenvalues on the open left half-plane and $\|T_{wd}\|_\infty \le \sigma$. Also, if $A - BKC$ is asymptotically stable, one can test $\|T_{wd}\|_\infty \le \sigma$ by computing the eigenvalues of the Hamiltonian matrix

$$H_K = \begin{bmatrix} A - BKC & DD' \\ -E'E & -A' + C'K'B' \end{bmatrix}$$

It is known that $\|T_{wd}\|_\infty \le \sigma$ if and only if H_K has no eigenvalues on the imaginary axis. The details can be found in Ref. 2.

IV. Illustrative Examples

The first example is the F100 turbofan engine model.[12]

Example 1

The parameters of the 16th order state-space model of the F100 turbofan engine model with five inputs and five outputs are

$$A = \left[\begin{array}{rrrrrrrr}
-4.328 & 0.1714 & 5.376 & 401.6 & -724.6 & -1.933 & 1.020 & -0.9820 \\
-0.4402 & -5.643 & 127.5 & -233.5 & -434.3 & 26.59 & 2.04 & -2.592 \\
1.038 & 6.073 & -165.0 & -4.483 & 1049 & -82.45 & -5.314 & 5.097 \\
0.5304 & -0.1086 & 131.3 & -578.3 & 102.0 & -9.240 & -1.146 & -2.408 \\
0.00848 & -0.01563 & 0.0560 & 1.573 & -10.05 & 0.1952 & -0.0088 & -0.0211 \\
0.8350 & 0.0125 & -0.0357 & -0.6074 & 37.65 & -19.79 & -0.1813 & -0.0296 \\
0.6768 & -0.0126 & -0.0968 & -0.3567 & 80.24 & -0.0824 & -20.47 & -0.0393 \\
-0.0970 & 0.8666 & 16.87 & 1.051 & -102.3 & 29.66 & 0.5943 & -19.97 \\
-0.0088 & -0.0164 & 0.1847 & 0.2169 & -8.420 & 0.7003 & 0.0567 & 6.623 \\
-0.0001 & -0.0002 & 0.0027 & 0.0032 & -0.1246 & 0.0104 & 0.0008 & 0.0981 \\
-1.207 & -6.717 & 26.26 & 12.49 & -1269.0 & 103.0 & 7.480 & 36.84 \\
-0.0273 & -0.4539 & -52.72 & 198.8 & -28.09 & 2.243 & 0.1794 & 9.750 \\
-0.0012 & 0.0202 & -2.343 & 8.835 & -1.248 & 0.0998 & 0.0081 & 0.4333 \\
-0.1613 & -0.2469 & -24.05 & 23.38 & 146.3 & 1.638 & 0.1385 & 4.486 \\
-0.0124 & 0.0302 & -0.1198 & -0.0482 & 5.675 & -0.4525 & 19.81 & 0.1249 \\
-1.653 & 1.831 & -3.822 & 113.4 & 341.4 & -27.34 & -2.040 & -0.6166
\end{array}\right.$$

$$\left.\begin{array}{rrrrrrrr}
0.9990 & 1.521 & -4.062 & 9.567 & 10.08 & -0.6017 & -0.1312 & 0.0960 \\
11.32 & 10.90 & -4.071 & -0.0574 & -0.6063 & -0.0749 & -0.5936 & -0.0960 \\
-0.0094 & 0.1352 & 5.638 & 0.0225 & 0.1797 & 0.0241 & 1.100 & 0.0274 \\
-3.081 & -4.529 & 5.707 & -0.2346 & -2.111 & -0.2460 & -0.4686 & -0.3223 \\
0.0021 & -0.0526 & -0.0408 & -0.0092 & -0.0818 & 0.0343 & 0.0050 & -0.0126 \\
-0.0195 & -0.1622 & -0.0064 & -0.0235 & -0.2201 & -0.0251 & -0.0037 & -0.0336 \\
0.0188 & -0.2129 & -0.0093 & -0.0314 & -0.2919 & -0.0337 & 0.0887 & -0.0446 \\
0.0225 & 0.1791 & 0.0084 & 0.0265 & 0.2560 & 0.0284 & -0.0375 & 0.0365 \\
-49.99 & 0.0676 & 39.46 & 0.0050 & 0.0898 & 0.0053 & 0.000 & 0.0137 \\
-0.6666 & -0.6657 & 0.5847 & 0.67e-4 & 0.0013 & 0.71e-4 & 0.000 & 0.21e-3 \\
0.2854 & 2.332 & -47.65 & 0.3406 & 3.065 & 0.3624 & -0.4343 & 0.4681 \\
-9.627 & -9.557 & 38.48 & -50.01 & 0.1011 & 0.0120 & -0.0469 & 0.0172 \\
-0.4278 & -0.4245 & 1.710 & -2.000 & -1.996 & 0.53e-3 & -0.0020 & 0.75e-3 \\
-4.414 & -4.354 & 17.66 & -3.113 & -3.018 & -19.77 & -0.0500 & 0.0151 \\
-0.0011 & -0.0068 & 0.0184 & -0.0010 & -0.0135 & -0.0012 & -20.00 & -0.0021 \\
0.5004 & 0.1437 & -2.416 & -0.1073 & -1.078 & 30.63 & 19.89 & -50.16
\end{array}\right]$$

$$B = \left[\begin{array}{rrrrr}
-0.0457 & -451.6 & -105.8 & -1.506 & 851.5 \\
0.1114 & -546.1 & -6.575 & -107.8 & 3526 \\
0.2153 & 1362 & 13.46 & 20.14 & -6777 \\
0.3262 & 208.0 & -2.888 & -1.653 & -269.1 \\
0.0099 & -98.39 & 0.5069 & -0.1940 & -94.70 \\
0.0273 & 71.62 & 9.608 & -0.3160 & -184.1 \\
0.0172 & 71.71 & 8.571 & 0.7989 & -515.2 \\
-0.0774 & -141.2 & -0.8215 & 39.74 & 1376 \\
0.0386 & -7.710 & -0.0437 & -0.1024 & -6684 \\
0.0006 & -0.1144 & -0.64e-3 & -0.0014 & -99.02 \\
5.727 & -1745 & -8.940 & -17.96 & 88980 \\
0.1392 & -24.30 & -0.2736 & -0.3403 & -6931 \\
0.0062 & -1.082 & -0.0118 & -0.0145 & -307.7 \\
0.0678 & 16.60 & 0.3980 & 0.0231 & -2588 \\
0.0019 & 9.147 & -0.8241 & 0.0898 & -32.31 \\
0.1677 & 435.8 & -89.94 & 4.900 & -295.5
\end{array}\right]$$

and

$$C^T = \begin{bmatrix} 0.4866 & 0.1383 & 0.0000 & 0.74e-4 & -0.15e-4 \\ -0.6741 & 0.2789e-5 & 0.0000 & 0.55e-5 & -0.12e-3 \\ 5.392 & 0.0000 & 0.0000 & 0.48e-5 & 0.25e-2 \\ 95.42 & 0.0000 & 0.0000 & 0.15e-3 & 0.16e-3 \\ 24.03 & -0.0108 & 0.0000 & -0.0150 & -0.0162 \\ 10.52 & -0.55e-4 & 0.0000 & -0.65e-4 & 0.0011 \\ 0.8190 & 0.47e-4 & 0.0000 & 0.88e-4 & 0.96e-4 \\ -0.4492 & 0.0000 & 0.0000 & 0.50e-5 & 0.55e-5 \\ 0.5195 & 0.0000 & 0.0000 & 0.34e-5 & 0.37e-5 \\ 0.8437 & 0.0000 & 0.0000 & 0.27e-4 & 0.30e-4 \\ -1.863 & 0.0000 & 1.000 & 0.11e-5 & 0.12e-5 \\ 0.0571 & 0.0000 & 0.0000 & 0.4e-5 & 0.44e-5 \\ 0.4815 & 0.0000 & 0.0000 & 0.37e-4 & 0.40e-4 \\ 3.428 & 0.0000 & 0.0000 & 0.43e-5 & 0.47e-5 \\ 2.161 & 0.0000 & 0.0000 & 0.50e-5 & 0.53e-5 \\ 0.768 & 0.0000 & 0.0000 & 0.56e-5 & 0.61e-5 \end{bmatrix}$$

It can be verified that the preceding physical plant satisfies assumptions 1–3. We are interested in minimizing sensitivity function $\mathcal{S}(s) = [I + KP(s)]^{-1}$ in the H^∞ norm sense. It is clear that by the strict properness of the plant, $\|\mathcal{S}\|_\infty \geq 1$. It will be demonstrated that for any $\gamma > 1$ there exists an internally stabilizing static controller K such that $\|\mathcal{S}\|_\infty < \gamma$. To proceed, we note that

$$\mathcal{S}(s) = [I + KP(s)]^{-1} = I - C(sI - A + BKC)^{-1}B$$

and hence

$$\|\mathcal{S}\|_\infty \leq 1 + \|\tilde{\mathcal{S}}\|_\infty, \qquad \tilde{\mathcal{S}}(s) = C(sI - A + BKC)^{-1}B$$

This result implies that the H^∞ optimization problem is in a matched case:

$$E = C, \qquad D = B \tag{20}$$

$\sigma > 0$, we can follow the procedure in Theorem 4 to seek the static controller K such that $A - BKC$ is stable and $\|\tilde{\mathcal{S}}\|_\infty < \sigma$. This fact reaffirms the earlier claim that for any $\gamma > 1$, an internally stabilizing static controller exists such that the sensitivity function $\mathcal{S}(s)$ has H^∞ smaller than γ. We will synthesize a static controller such that the closed-loop system is internally stable and $\|\mathcal{S}\|_\infty \geq 1.025$. This result is true if $\|\tilde{\mathcal{S}}\|_\infty \leq \sigma = 0.025$.

Following the procedure in the proof of Theorem 4, the solution S_{22} is obtained from Eqs. (15–17) with $\Phi_{22} = I_{11}$, $\hat{E} = 0$, $L = I_{11}$, and $\sigma = 0.025$. Knowing S_{22}, S_{11}, and S_{22} can be computed from

$$S_{12} = -C_1^{-1}C_2S_{22} \quad \text{and} \quad S_{11} = I_5 + S_{12}S_{22}^{-1}S_{21}$$

The preceding S is clearly positive definite. We also obtain

$$\hat{K} = (C_1S_{11} + C_2S_{21})^{-1} = \begin{bmatrix} 0.0092 & 1.0882 & 0.1332 & -151.94 & 36.591 \\ 0.0000 & -0.0078 & -0.0000 & 0.5883 & -0.0310 \\ 0.0000 & -0.6618 & -0.0001 & -2.0586 & -0.1519 \\ -0.0037 & 0.4998 & 0.0134 & 21.752 & 19.447 \\ -0.0000 & -0.0002 & 0.0000 & 0.0255 & 0.0009 \end{bmatrix}$$

As the B matrix is not in the form of Eq. (2), we compress the rows of B to get

$$B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

with B_1 square and nonsingular and set the similarity transform as

$$T = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & I_{11} \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} B_1 & 0 \\ 0 & I_{11} \end{bmatrix}$$

Hence, by setting $\{A, B, C\} = \{TAT^{-1}, TB, CT^{-1}\}$, the input matrix B is now in the form of Eq. (2). We can now partition A, C, and E as in Theorem 4, and verify that the matrix $F = A_{22} - A_{21}C_1^{-1}C_2$ is stable. Hence, the preceding system is strict minimum phase. It is now easy to verify that assumptions 1–3 are true. The matched condition (20) implies that $R(s)$ in Theorem 4 is identically zero. Hence, for any

by Eq. (18). Finally, by setting $K = (1/2\epsilon)\hat{K}$, we find that with $\epsilon = 0.2439e - 7$, the closed-loop system is internally stable and admits disturbance attenuation $\|\tilde{\mathcal{S}}\|_\infty \leq 0.0221 < \sigma$. This result concludes that the sensitivity function satisfies $\|\mathcal{S}\|_\infty < 1.0221$.

Example 2

In this example we show how systems that do not satisfy all of assumptions 1–3 may be handled. The system in this case does not satisfy the condition that $\det(CB) \neq 0$. However, for this particular system, it was found that if we obtained a lower-order approximation, it was possible to make the system satisfy assumptions 1–3. Clearly, the controller will be designed for the lower-order system, and one needs to verify whether the compensator achieves the required performance on the original full-order system.

Consider the fifth-order system with two inputs and outputs. The following parameters represent the state-space model of a drum boiler[1]

$$A = \begin{bmatrix} -0.129 & 0 & 0.0396 & 0.025 & 0.0191 \\ 0.00329 & 0 & -0.0000779 & 0.000122 & -0.621 \\ 0.0718 & 0-0.1 & 0.000887 & -3.85 & \\ 0.0411 & 0 & 0 & -0.0822 & 0 \\ 0.000361 & 0 & 0.000035 & 0.0000426 & -0.0743 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0.00139 \\ 0 & 0.0000359 \\ 0 & -0.00989 \\ 0.0000249 & 0 \\ 0 & -0.00000534 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

It can be verified that the transmission zeros of the system are all on the strict left half-plane although the system itself is not asymptotically stable. Hence, assumptions 1 and 3 are true, but $\det(CB) = 0$, which violates assumption 2. Although assumption 2 does not hold, one may decompose the system into sum of stable and antistable subsystems and perform model reduction on the stable subsystem. The Hankel singular values of the stable part are

$$\{1.6949e-02 \quad 4.5444e-03 \quad 2.6364e-04 \quad 2.9596e-06\}$$

which indicates that the stable subsystem can be well approximated by a third-order one using a balanced model reduction procedure. After model reduction and suitable similarity transformation, we have a new reduced-order model

$$\hat{A} = \begin{bmatrix} 2.1801e-01 & -1.73e+02 & 2.96e+02 & 0 \\ 6.32e-04 & -4.59e-01 & 8.82e-01 & 0 \\ 9.97e-06 & -5.32e-03 & -6.11e-02 & 0 \\ 7.03e-09 & -4.98e-06 & 9.89e-06 & 0 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\hat{C} = \begin{bmatrix} -2.9755e-07 & 1.3904e-03 & -2.8272e-03 & 2.5023e-14 \\ -8.3450e-08 & 3.6004e-05 & 5.6820e-03 & -8.2232 \end{bmatrix}$$

It is then clear that assumptions 1–3 hold true for reduced-order model $\{\hat{A}, \hat{B}, \hat{C}\}$. One can now synthesize the static feedback gain K for certain specific H^∞ optimization problems. One of them is the weighted sensitivity minimization: find K such that $A - BKC$ is asymptotically stable and $\| W_1(I + KP)^{-1} W_2 \|_\infty$ is minimized among all static feedback gain. Let $\{A_1, B_1, C_1\}$ and $\{A_2, B_2, C_2\}$ be realizations of the weighting functions $W_1(s)$ and $W_2(s)$. Then, an equivalent model for the weighted minimization problem is

$$A_e = \begin{bmatrix} \hat{A} & 0 & 0 \\ 0 & A_2 & 0 \\ B_1\hat{C} & B_1C_2 & A_1 \end{bmatrix} \qquad D_e = \begin{bmatrix} 0 \\ B_2 \\ 0 \end{bmatrix} \qquad B_e = \begin{bmatrix} \hat{B} \\ 0 \\ 0 \end{bmatrix}$$

and

$$E_e = \begin{bmatrix} 0 & 0 & C_1 \\ \hat{C} & C_2 & 0 \end{bmatrix} \qquad C_e = [\hat{C} \quad C_2 \quad 0]$$

It is easy to verify that for this equivalent system, assumptions 1–3 are satisfied. We can then follow the algorithm in Sec. III to synthesize the static gain K. Because of the space limit, we have not given detailed computation but would like to re-emphasize that as the approximant is used in weighted sensitivity minimization, one needs to verify the final results for the original system to insure the internal stability of the closed-loop system and the true disturbance attenuation level. Since the approximation error is very small, it can be predicted that a static feedback gain can be synthesized to achieve the internal stability of the true closed-loop system with reasonable disturbance attenuation level. The details are omitted.

V. Concluding Remarks

In this paper we studied the problem of disturbance attenuation and H^∞ optimization with constraint linear output feedback control. It was shown that for a class of linear multivariable systems it is possible to obtain achievable disturbance attenuation level with linear static output feedback. Based on the results developed in the paper, a computational procedure to achieve prespecified disturbance attenuation was developed.

Acknowledgments

This research was partially supported by National Science Foundation Grant ECS-9110636. The authors would like to acknowledge the constructive comments of the anonymous reviewers that led to considerable improvement in the presentation of the results.

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